NUMERICAL MODELING OF SPACE PLASMA FLOWS: ASTRONUM-2008
ASP Conference Series, Vol. 406, © 2009
Nikolai V. Pogorelov, Edouard Audit, Phillip Colella, and Gary P. Zank, eds.

Numerical Cosmic-Ray Hydrodynamics

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Abstract. We present a numerical method for integrating the equations describing a system made of a fluid and cosmic-rays. We work out the modified characteristic equations that include the CR dynamical effects in smooth flows. We model the energy exchange between cosmic-rays and the fluid, due to diffusive processes in configuration and momentum space, with a flux conserving method. For a specified shock acceleration efficiency as a function of the upstream conditions and shock Mach number, we modify the Riemann solver to take into account the cosmic-ray mediation at shocks without resolving the cosmic-ray induced substructure. A self-consistent time-dependent shock solution is obtained by using our modified solver with Glimm's method. Godunov's method is applied in smooth parts of the flow.

1. Introduction

The system of equations describing a one-dimensional nonrelativistic fluid coupled to suprathermal cosmic-ray particles (heretofore CR) through the exchange of momentum and energy reads

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{1}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}[\rho u^2 + P_g] = -\frac{\partial P_c}{\partial x}, \tag{2}$$

$$\frac{\partial \rho e_g}{\partial t} + \frac{\partial}{\partial x}[(\rho e_g + P_g)u] = -u\frac{\partial P_c}{\partial x} - \Sigma, \tag{3}$$

where (ρ, u, P_g, e_g) indicate the gas density, velocity, pressure and specific energy respectively, with $e_g = u^2/2 + e_{th}$ and e_{th} the specific thermal energy. A γ-law equation of state, $e_{th} = P_g/\rho(\gamma_g - 1)$, is assumed. The CR pressure, P_c, is defined through the CR distribution function $f(x,p,t)$ as

$$P_c(x,t) = \frac{4\pi}{3} m_c c^2 \int_{p_{min}}^{p_{max}} p^4 f(x,p,t)\,(p^2+1)^{-\frac{1}{2}}\,dp, \tag{4}$$

where p is momentum in units of '$m_c c$', with m_c the CR particle mass. 'f' evolves according to the diffusion-convection equation Skilling (1975)

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\left(\kappa \frac{\partial f}{\partial x}\right) = \frac{1}{3}\frac{\partial u}{\partial x} p \frac{\partial f}{\partial p} + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(b_\ell\, f + D_p \frac{\partial f}{\partial p}\right)\right], \tag{5}$$

where $\kappa(x,p)$ is the spatial diffusion coefficient, $b_\ell(p) \equiv -(dp/dt)_{loss}$ describes the particle momentum losses and $D_p(p)$ is the momentum diffusion coefficient.

Finally, the source term Σ in Eq. (3) describes the exchange of energy between CRs and the fluid Miniati (2007)

$$\Sigma(x) = \quad -4\pi m_c c^2 \int_{p_{min}}^{p_{max}} \left(b_{m\ell}\, f + D_p \frac{\partial f}{\partial p} \right) \frac{p^3}{(p^2+1)^{\frac{1}{2}}}\, dp$$
$$-4\pi m_c c^2 p^2 f \left(\frac{1}{3}\frac{\partial u}{\partial x} + b_\ell + D_p \frac{\partial \ln f}{\partial p} \right) \left[(p^2+1)^{\frac{1}{2}} - 1 \right] \Big|_{p=p_{min}}, \quad (6)$$

where $b_{m\ell}(p)$ includes mechanical losses only (i.e. radiative losses are excluded).

In smooth flows and on large enough scales, $\lambda \gg \lambda_{mfp} \sim \kappa(p)/c$ (e.g. $\lambda \simeq 0.01$pc for $\kappa(\mathrm{GeV/c}) \simeq 10^{27}\mathrm{cm}^2\mathrm{s}^{-1}$, Hartquist & Morfill 1986), the presence of CRs enhances the propagation speed of sound waves but simultaneously causes damping of their amplitude due to CR diffusion Parker (1965). In addition energy is exchanged non adiabatically between the thermal and nonthermal components according to the Σ term in Eq. (3). This term arises from diffusive processes and it seems plausible that as long as the relevant transport coefficients, κ and D_p, are correctly provided, it can be properly modeled numerically.

Around shocks the diffusion process gives rise to an efficient mechanism for transferring energy from the flow to the particles Drury (1983). In this case, in solving numerically the system of Eq. (1-5), a major difficulty arises due the large disparity between the microphysics scales where the exchange of momentum and energy between fluid and CR particles takes place, on the one hand, and the large macroscopic scales of astrophysical systems that one is interested in modeling, on the other. In fact the backreaction of the particles changes the shock's structure, jump conditions and propagation speed Achterberg et al. (1984); Malkov (1997), i.e. it has macroscopic consequences. However, the diffusion process responsible for such modification operates on scales that range from the shock thickness up to the diffusive scale length of the highest energy CR particles, $\lambda_\kappa(p_{max}) = \kappa(p_{max})/u_{shock}$, where u_{shock} is the shock speed. These microscopic scales cannot be resolved simultaneously with the large scales characterizing astrophysical systems. In addition, diffusive shock acceleration is characterized by a rich variety of complex plasma processes. Therefore, one quickly realizes that an *explicit* numerical treatment of this process makes sense only when specifically studying the physics of the shock acceleration process itself Ellison & Eichler (1984); Donohue & Zank (1993); Berezhko et al. (1994); Jones & Kang (2005). If on the other hand one is interested in the CR dynamical contribution on large astrophysical scales Miniati et al. (2001); Hanasz et al. (2004); Pfrommer et al. (2006), a simplified approach should be taken Miniati (2001); Enßlin et al. (2007), which allows to calculate with some degree of accuracy the source terms on the RHS of Eq. (1-3) and, likewise, modify the shock solution of the fluid equations for the effects due to the CR mediation. In this second approach one specifies as part of the simulation input parameters: (a) the CR transport coefficients in terms of the macroscopic thermodynamic properties of the fluid and (b) the shock acceleration efficiency and the accelerated CR distribution functions as a function of the upstream fluid conditions and the shock Mach number.

The two-fluid model provides the simplest way to include the CR effects described above. Here the CRs are described as a fluid characterized by a γ-law equation of state such that CR energy and pressure relate as $E_c = P_c/(\gamma_c - 1)$. Only the parameter γ_c needs to be specified. The acceleration efficiency is implicitly determined by the conservation equations Achterberg et al. (1984). While useful and applied with some success in order to study individual shocks, this approach may be too restrictive to study a general astrophysical source with supersonic motions. Particles accelerated at shocks with different Mach number will have different distribution functions and hence will define different local values of γ_c. This means that the efficiency obtained at fixed value of γ_c is too restrictive and, in fact, it can even lead to unphysical shock solutions Achterberg et al. (1984); Malkov (1997). In addition, in this approach no information about the particle distribution in momentum space is retained. Since the processes determining both Σ_c as well as the evolution of P_c are strongly momentum dependent, the above lack of information can lead to inaccurate estimates of the RHS of Eq. (2-3).

To obviate such issues we propose an approach that, while allowing a fluid-like description of the CR component, retains the essential information at the kinetic level to avoid the above pitfalls. This is briefly described in the following.

2. A Scheme for Cosmic-Ray Hydrodynamics

In our approach we rewrite the system (1-5) in fully conservative form, so that energy and momentum contain both the fluid and CR components, and the only term on the RHS is due to radiative energy losses. The effects of CRs pressure on the hydrodynamics are accounted for by a modified formulation of the fluxes. Similarly, having specified the parameters defining the shock acceleration process (item (b) above), the CR effects on the shock solution are accounted for by a modified Riemann solver, even though the shock structure is not resolved. The proposed Riemann solver works most easily when applied to the full shock jump conditions, not to intermediate shock jumps created as a result of numerical viscosity. We have therefore implemented it together with a simple shock tracking scheme provided by Glimm's method Colella (1982).

To cost-effectively retain information about the shape of the CR distribution function, we divide momentum space into a few (~ 10) log-spaced coarse *bins* and assume that the distribution function f within each bin is a power-law in momentum. For convenience, we store the volume integral of 'f' within each bin, namely $n_{p_j} = \int_{p_{j-\frac{1}{2}}}^{p_{j+\frac{1}{2}}} 4\pi p^2 f(p) dp$. A reconstruction scheme then allows us to recover the piece-wise power-law distribution function from the set $\{n_{p_j}\}$ Miniati (2001). The sub-bin power-law model turns out an effective way to compensate for the fewness of resolution elements available in momentum space.

2.1. Finite Difference Diffusion-Convection Equation

Integrating the diffusion-convection equation (5) multiplied by $4\pi p^2$ over each bin we obtain an equation for the time evolution of each n_{p_j}. This equation can be used to derive finite-volume scheme for n_{p_j} Miniati (2001)

$$n_{p_j}^{t+\Delta t} - n_{p_j}^{t} = -\frac{\Delta t}{\Delta p}\left(F_{p_{j+\frac{1}{2}}}^{n+\frac{1}{2}} - F_{p_{j-\frac{1}{2}}}^{n+\frac{1}{2}}\right) - \Delta t\left[\left(F_{x_{i+\frac{1}{2}}}^{n+\frac{1}{2}} - F_{x_{i-\frac{1}{2}}}^{n+\frac{1}{2}}\right) + J_{p_j}\right], \quad (7)$$

where the fluxes are time-averages evaluated at momentum and spatial cell interfaces, respectively, and J_{p_j} is an additional term for the production rate of CRs due to shock acceleration. The fluxes are defined as

$$F_x = u\, n_{p_j} - \langle\kappa\rangle \nabla n_{p_j}, \quad F_p = 4\pi p^2 f(p) \left(-\frac{1}{3}\frac{\partial u}{\partial x} p - b_\ell(p) - D_p \frac{\partial \ln f}{\partial p} \right), \quad (8)$$

accounting for both spatial advection and diffusion (F_x) on the one hand, and adiabatic effects, energy losses and diffusion in momentum space (F_p), on the other. In Eq. (8) $\langle\kappa\rangle$ is the ∇f-weighted average of $\kappa(p)$ in momentum volume. The flux in momentum space at time $t = n\Delta t$ is computed as Miniati (2001)

$$F^n_{p_{j+\frac{1}{2}}} = \frac{\Delta p}{\Delta t} \int_{p_{j+\frac{1}{2}}}^{p_u} 4\pi p^2 f^n(p) dp, \quad \Delta t = \int_{p_{j+\frac{1}{2}}}^{p_u} \frac{dp}{-\frac{1}{3}\frac{\partial u}{\partial x} p - b_\ell(p) - D_p \frac{\partial \ln f}{\partial p}}, \quad (9)$$

where, in the first equation above, the distribution function is evaluated left or right of the interface $p_{j+1/2}$, depending on whether $p_u \leq p_{j+1/2}$ or $p_u > p_{j+1/2}$, respectively. The quantity p_u defined by the characteristic (second) equation represents the momentum that particle must have at time t, in order to have a momentum $p_{j+1/2}$ an interval Δt later. Note that the diffusive term in the above expression for the flux can be retained only as long as diffusion is slow, i.e. $(p_{j+1/2} - p_{j-1/2})^2/D_p \gg \Delta t$. Time centering of the flux in momentum space ($F^n_p \rightarrow F^{n+\frac{1}{2}}_p$) is obtained by time averaging between t and $t + \Delta t$, as usually done for nonstiff sources. Time centering is required because p_u depends on the time dependent fluid properties.

2.2. Spatial Fluxes and Modified Riemann Solver

In this and next subsection we ignore energy losses (except adiabatic ones), which are addressed in Miniati (2001), and diffusion. Diffusion is naturally included in our scheme (Eq. 8) and, depending on its stiffness, can be readily treated with either implicit or explicit schemes available in the literature (e.g. Miniati & Colella 2007). Instead, in the following we focus on reproducing the correct CR modified shock structure, including the jump conditions and shock speed, without explicitly resolving the diffusive scales of the CRs.

In order to compute the spatial fluxes, F_x, one needs to analyze the hyperbolic structure of the fluid+CR system. The quantities n_{p_j} are advected passively, so it is sufficient to consider the system with primitive variables (ρ, u, P_g, P_c). The system has four left/right eigenvectors (not specified here but see, Miniati 2007) associated to the following eigenvalues: $\lambda_0 = u - c_s$, $\lambda_1 = u$, $\lambda_2 = u$, $\lambda_3 = u + c_s$, where $c_s = \sqrt{(\gamma_g P_g + \gamma_c P_c)/\rho}$ is the modified sound speed that accounts for the CR pressure. The corresponding characteristic equations include the usual relations for an ordinary gas but with gas pressure and sound speed replaced by $P = P_g + P_c$ and c_s, respectively, and the additional relation $dP_c/\gamma_c P_c = dP_g/\gamma_g P_g$ describing the change of CR pressure as a function of the gas pressure during an adiabatic process. In smooth flows this information is sufficient to obtain the time averaged spatial fluxes. In fact, one can readily apply a Godunov-type scheme, after including the above simple modifications, to compute the intermediate states at cell interfaces which define the fluxes.

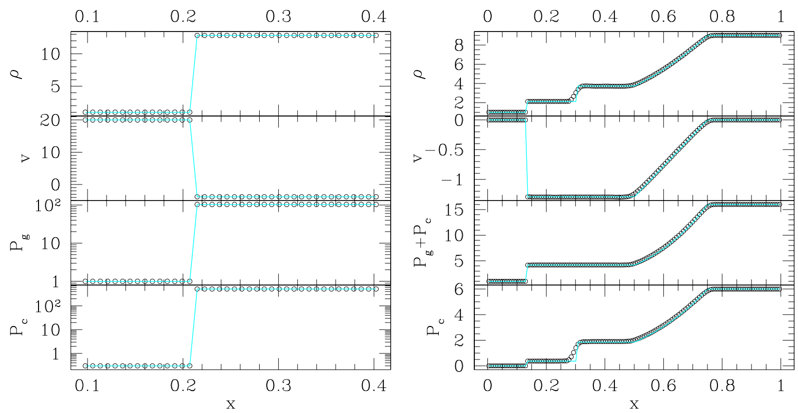

Figure 1. Numerical (open circles) versus exact (solid line) solutions for a left moving shock with Mach number 20 (left) and a shock tube problem (right). Shocks are advanced with Glimm's method and smooth flows with Godunov's method

When a shock is present, however, the Riemann solver procedure needs to be modified. The first step in a Riemann solver is to compute the velocity, u^*, and pressure, P^*, of the central state separating the left and right states. The central state depends on the speed of the nonlinear waves. When CR acceleration affects the shock structure the Lagrangian speed of the nonlinear waves takes the modified form Miniati (2007)

$$W = C_g^- \left[\frac{2 r_p^{\gamma_g}}{\gamma_g + 1 - r_p^{-1}(\gamma_g - 1)} \left(1 + \frac{\gamma_g + 1}{2\gamma_g \, r_p^{\gamma_g}} \frac{P^* - P^-}{P_g^-} \right) \right]^{\frac{1}{2}}, \tag{10}$$

where, $\pm$, labels values upstream and downstream of the shock, respectively, $C_s = \sqrt{\gamma_g \rho P_g}$ is the Lagrangian sound speed of an ordinary gas and, most importantly, r_p is the adiabatic compression of the fluid produced by the energetic CR particles as they diffuse upstream of the shock Achterberg et al. (1984). Provided the shock acceleration efficiency as a function of the upstream conditions and the shock Mach number, using Euler's equation one obtains an implicit expression for the nonlinear wave speed, namely, $W = W\left[P^*, r_p\left(\frac{W}{C_g}\right)\right]$, which is solved iteratively within the Riemann solver. In addition, the tangent slopes to the wave curves in the $P\!-\!u$ plane connecting the left and right states, which are used to find P^*, also need to be modified (see details in, Miniati 2007). These modifications account for the changes in the shock speed and jump conditions due to CR mediation, consistently with the assumed shock acceleration model. The Riemann solution allows the calculation of the spatial fluxes and the energy dissipated into CRs. The distribution function of shock accelerated particles from the input model parameters then specifies the source term J in Eq. (7).

2.3. Numerical Tests

To illustrate the performance of our scheme in the following we present two tests. In both cases the initial conditions consist of a Riemann problem with constant left and right states specified by the following quantities $(\rho, u, P_g, P_c, \gamma_c)$. We deliberately omit a description of the distribution function to focus on the thermodynamic properties of the fluid. For simplicity the shock acceleration is assumed independent of the upstream condition and dependent solely on the shock Mach number as, $\eta(\mathcal{M}) = 0.8[1 - \exp(\mathcal{M} - 1.5)/5.77]$, where η is the fraction of total momentum upstream of the shock that is converted into downstream CR pressure. The first test consists of a strong shock moving to the left with left/right initial states: $(1.0, 20, 1.0, 0.3, 1.34)$ and $(12.83, -3.81, 103.28, 512.73, 1.33)$, respectively. The second test is a shock tube problem with left with left/right initial states $(1, 0, 1, 0, N/A)$ and $(9, 0, 10, 6, 1.33)$, respectively. The results for each test are shown in the left and right plots of Fig. (1), respectively. For each plot the four panels show, from top to bottom, gas density, velocity, gas pressure or total pressure, and CR pressure. In general the numerical solutions reproduce the 'exact' solutions very well, without oscillations or artifacts, even when the CR pressure is comparable or significantly higher than the thermal pressure. Note that in this demonstration we have neglected all losses except adiabatic ones for clarity. As a result, for a lagrangian fluid elements the CR distribution function has a constant slope determined by the fluid element position with respect to the contact discontinuity.

3. Conclusions

We have presented a one dimensional method to follow efficiently the evolution of the CR distribution function in large scale astrophysical systems, and to include the dynamical effects of CRs both in smooth flows and at shocks. Multidimensional extension of the current scheme will be part of future work.

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